Statistical Mechanics - 2019-2 - Problem set 5

Professor: Gabriel T. Landi

Deadline: 10/12

1. **Entropy of non-interacting systems:** Consider a system described by the non-interacting Hamiltonian
$$\mathcal{H} = \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha, \tag{1}$$
and placed in the grand-canonical state
$$\rho = \frac{e^{-\beta(\mathcal{H}-\mu\hat{N})}}{Q}. \tag{2}$$
Show that in the case of Fermions the von Neumann entropy is given by
$$S = -\sum_\alpha \Big[\bar{n}_\alpha \ln \bar{n}_\alpha + (1-\bar{n}_\alpha)\ln(1-\bar{n}_\alpha)\Big], \qquad \bar{n}_\alpha = \frac{1}{e^{\beta(\varepsilon_\alpha-\mu)}+1}, \tag{3}$$
whereas in the case of Bosons it is given by
$$S = -\sum_\alpha \Big[\bar{n}_\alpha \ln \bar{n}_\alpha - (1+\bar{n}_\alpha)\ln(1+\bar{n}_\alpha)\Big], \qquad \bar{n}_\alpha = \frac{1}{e^{\beta(\varepsilon_\alpha-\mu)}-1}. \tag{4}$$

2. **Ultra-relativistic Fermi gas:** Consider a gas of non-interacting particles with dispersion relation $\varepsilon_k = \hbar c k$, where $k = |\boldsymbol{k}|$ and c is the speed of light. Assume that the gas is d-dimensional and has spin S.
   (a) Compute the density of states. Discuss how it behaves for $d = 1, 2$ and 3.
   (b) Compute the Fermi energy ε_F.
   (c) Compute the ground-state energy. Leave your results solely in terms of ε_F, d and N.

3. **Bose-Einstein condensation for a generic density of states:** Consider a Bose gas with a density of states of the form
$$D(\varepsilon) = \Lambda_\eta \varepsilon^\eta, \tag{5}$$
where η is some generic exponent and Λ_η is a constant. Assume $\eta > -1$. The case we studied in class had $\eta = 1/2$.
   (a) Show that Bose-Einstein condensation at finite temperature will only take place if $\eta > 0$.
   (b) Assuming $\eta > 0$, find the critical temperature T_c. You can leave your result in terms of an integral. But the integral has to be a function only of η and no other parameter.
   (c) Compute the temperature dependence of the condensate fraction N_0/N below T_c.