1 Gás ideal ultra-relativistica em 2D

(a) De acordo com a fórmula de Gibbs, a prob. de encontrar o sistema no microestado (x, y, p_x, p_y) será

$$P(x, y, p_x, p_y) = \frac{e^{-\beta E}}{Z_1} \tag{5.1}$$

onde

$$Z_1 = \int dx\, dy\, dp_x\, dp_y\; e^{-\beta E} = A \int dp_x\, dp_y\; e^{-\beta c |p|}$$

Agora mudamos para coordenadas circulares

$$p_x = p \cos\theta$$
$$p_y = p \sin\theta$$

Com isso obtemos

$$Z_1 = A \int_0^\infty dp \int_0^{2\pi} d\theta\; p\, e^{-\beta c p} = A\, 2\pi \int_0^\infty p\, e^{-\beta c p}\, dp$$

$$= -A\, 2\pi \frac{\partial}{\partial(\beta c)} \int_0^\infty e^{-\beta c p}\, dp = -2\pi A \frac{\partial}{\partial(\beta c)} \frac{1}{(\beta c)}$$

Portanto

$$Z_1 = \frac{2\pi A}{(\beta c)^2} \tag{5.2}$$

Assim, a distribuição de Gibbs será

$$\underline{P}(x, y, p_x, p_y) = \frac{1}{A} \frac{(\beta c)^2}{2\pi} e^{-\beta c\sqrt{p_x^2 + p_y^2}} \tag{5.3}$$

As variáveis (x, y) e (p_x, p_y) são independentes, pois a dist. de probabilidades se fatora na forma de um produto

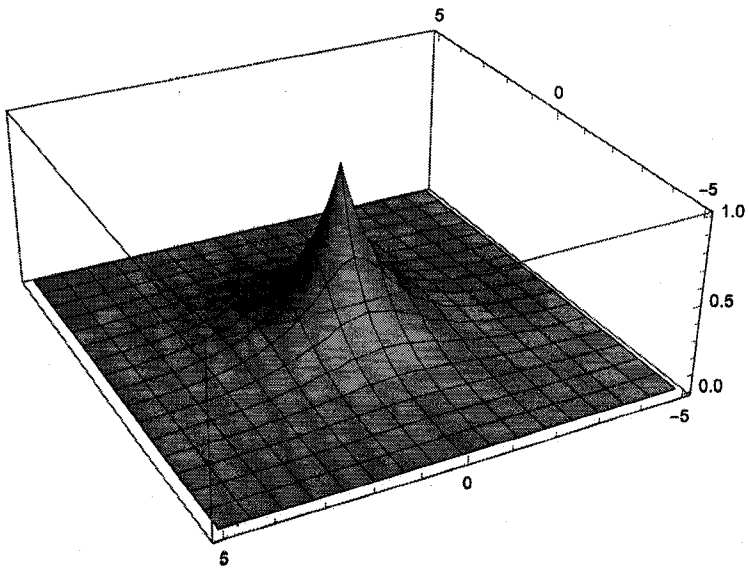
$$\underline{P}(x, y, p_x, p_z) = \underline{P}(x, y)\, \underline{P}(p_x, p_z) \tag{5.4}$$

com

$$\underline{P}(x, y) = \frac{1}{A} \qquad \text{(Uniforme em um quadrado)}$$

$$\underline{P}(p_x, p_y) = \frac{(\beta c)^2}{2\pi} e^{-\beta c\sqrt{p_x^2 + p_y^2}} \tag{5.5}$$

No entanto, p_x e p_y não são independentes pois $\underline{P}(p_x, p_y)$ não pode ser fatorado na forma de um produto.



(b) $\langle |p| \rangle = \int dp_x dp_y \, |p| \, \frac{(\beta c)^2}{2\pi} \, e^{-\beta c |p|}$

$$= \frac{(\beta c)^2}{2\pi} \, 2\pi \int_0^\infty dp \; p^2 \, e^{-\beta c p}$$

$$= (\beta c)^2 \, \frac{\partial^2}{\partial (\beta c)^2} \, \frac{1}{(\beta c)}$$

$$= \frac{2}{\beta c}$$

Portanto

$$\boxed{\langle |p| \rangle = \frac{2 k_B T}{c}} \qquad (5.6)$$

O momento médio aumenta linearmente com a temperatura.

(c) Como elas ~~são~~ são interagentes

$$\boxed{Z_N = Z_1^N = \left[\frac{2\pi A}{(\beta c)^2} \right]^{N!}} \qquad (5.7)$$

A energia interna vale

$$U = -\frac{\partial}{\partial \beta} \ln Z_N = -\frac{\partial}{\partial \beta} \left[\ln \beta^{2N} \right] = \frac{2N}{\beta}$$

$$\therefore \quad \boxed{\begin{aligned} U &= 2 N k_B T \\ C &= \frac{\partial U}{\partial T} = 2 N k_B \end{aligned}} \qquad (5.8)$$

(e) $Z_1 = V \int d^3p \, e^{-\beta c p} = V \, 4\pi \underbrace{\int_0^\infty dp \, p^2 e^{-\beta c p}}_{\frac{2}{c^2\beta^3}}$

$$\therefore \quad Z = \left(\frac{8\pi V}{c^2 \beta^3} \right)^N$$

$$U = -\frac{\partial}{\partial \beta} \ln Z = N \frac{\partial}{\partial \beta} (3 \ln \beta) = \frac{3N}{\beta}$$

$$\therefore \quad U = 3 N k_B T$$

$$C = \frac{\partial U}{\partial T} = 3 N k_B$$

A pressão será

$$P = -\frac{\partial F}{\partial V} = N k_B T \frac{\partial}{\partial V} \ln V$$

$$\therefore \quad P = \frac{N k_B T}{V}$$

2) $$Z = \left(\frac{V-Nb}{N}\right)^N \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3N/2} \exp\left(\frac{N^2a}{Vk_BT}\right)$$

(a) $$p = -\frac{\partial F}{\partial V}$$

$$= k_BT \frac{\partial}{\partial V}\left\{N \ln\left(\frac{V-Nb}{N}\right) + \frac{N^2a}{Vk_BT}\right\}$$

$$= k_BT \frac{N}{V-Nb} - k_BT \frac{N^2a}{V^2k_BT}$$

$$\therefore \boxed{p = \frac{Nk_BT}{V-Nb} - \frac{N^2a}{V^2}}$$

(b) $$U = -\frac{\partial}{\partial\beta}\ln Z = -\frac{3N}{2}\frac{\partial}{\partial\beta}\ln(1/\beta) - \frac{\partial}{\partial\beta}\left(\frac{\beta N^2a}{V}\right)$$

$$\therefore \boxed{U = \frac{3}{2}Nk_BT - \frac{N^2a}{V}}$$

Definindo $u = U/N$ e $v = V/N$ obtemos

$$\boxed{u = \frac{3}{2}k_BT - \frac{a}{v}}$$

o que mostra que U é extensivo.

3) $\rho(\theta,\phi) = \frac{1}{A} \sin\theta \; e^{-\beta E}$

$$E = -\vec{\mu} \cdot \vec{\mathcal{E}} \qquad \vec{\mathcal{E}} = \mathcal{E}\,\hat{z}$$

$$\therefore \quad E = -\mu \mathcal{E} \cos\theta$$

$$\rho = \frac{1}{A} \sin\theta \; e^{\mu\beta\mathcal{E}\cos\theta}$$

(a) Normalização:

$$\int d\theta\, d\phi\, \rho = \frac{1}{A} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; e^{\mu\beta\mathcal{E}\cos\theta}$$

$$= \frac{1}{A}\, 2\pi \int_{-1}^{1} d(\cos\theta)\; e^{\mu\beta\mathcal{E}\cos\theta}$$

$$= \frac{1}{A}\, 2\pi \; \frac{e^{\beta\mathcal{E}\mu} - e^{-\beta\mathcal{E}\mu}}{\beta\mathcal{E}\mu}$$

$$= \frac{4\pi}{A} \; \frac{\sinh \beta\mathcal{E}\mu}{\beta\mathcal{E}\mu} = 1$$

$$\therefore \quad A = \frac{4\pi \sinh \beta\mathcal{E}\mu}{\beta\mathcal{E}\mu} \; //$$

(b) $P = \mu \langle \cos\theta \rangle$

$$= \mu \frac{\int d\theta\, d\phi \cos\theta \sin\theta\, e^{\mu\beta E\cos\theta}}{\int d\theta\, d\phi \sin\theta\, e^{\mu\beta E\cos\theta}}$$

$$= \mu \frac{\partial}{\partial(\beta E\mu)} \ln \int d\phi\, d\theta \sin\theta\, e^{\mu\beta E\cos\theta}$$

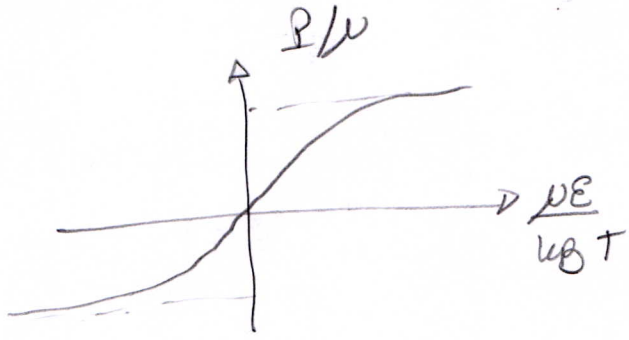
$$= \mu \frac{\partial}{\partial(\beta E\mu)} \ln A$$

$$\therefore \quad P = \mu \frac{\partial}{\partial(\beta E\mu)} \left[\ln \sinh \beta E\mu - \ln \beta E\mu \right]$$

$$= \mu \left[\coth \beta E\mu - \frac{1}{\beta E\mu} \right]$$

$$\boxed{P = \mu \left\{ \coth(\beta E\mu) - \frac{1}{\beta E\mu} \right\}}$$

(c)



(d) Para x pequeno

$$\coth(x) \simeq \frac{1}{x} + \frac{x}{3}$$

Portanto

$$P \simeq \mu \left\{ \frac{1}{\beta\mu\mathcal{E}} + \frac{\beta\mu\mathcal{E}}{3} - \frac{1}{\beta\mu\mathcal{E}} \right\}$$

$$\Rightarrow \quad P \simeq \frac{\mu^2 \mathcal{E}}{3 k_B T}$$

A susceptibilidade a campo nulo será, portanto

$$\chi = \frac{\partial P}{\partial \mathcal{E}} = \frac{\mu^2}{3 k_B T} \quad //$$

4) (a) Configuracões

1 em A, 2 em A : $E_1 = \Delta$

1 em A, 2 em B : $E_2 = \varepsilon$

1 em B, 2 em A : $E_3 = \varepsilon$

1 em B, 2 em B $E_4 = 2\epsilon + \Delta$

(b) $Z = \sum_m e^{-\beta E_m} = e^{-\beta\Delta} + 2e^{-\beta\epsilon} + e^{-\beta(2\epsilon+\Delta)}$

(c) $P_1 = \frac{e^{-\beta\Delta}}{Z}$ $P_2 = P_3 = \frac{e^{-\beta\epsilon}}{Z}$ $P_4 = \frac{e^{-\beta(2\epsilon+\Delta)}}{Z}$

(d) $U = \sum_m E_m P_m = \frac{1}{Z}\left\{\Delta e^{-\beta\Delta} + 2\epsilon e^{-\beta\epsilon} + (2\epsilon+\Delta) e^{-\beta(2\epsilon+\Delta)}\right\}$

(e) $S = \frac{U - F}{T} = -\frac{1}{T}\frac{\partial}{\partial\beta}\ln Z + k_B \ln Z$

$\therefore \frac{S}{k_B} = \ln Z - \beta\frac{\partial}{\partial\beta}\ln Z$ //

5) $k = 1, 0, -1$

$\varepsilon_1 = \varepsilon_{-1} > 0$

$\varepsilon_0 = 0$

(a) $Z = 1 + 2e^{-\beta\varepsilon}$

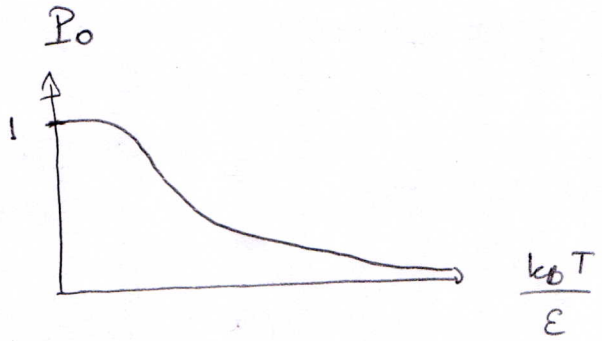
(b) $P_0 = \frac{1}{Z} = \frac{1}{1+2e^{-\beta\varepsilon}}$

Se $T \to 0$ então $\beta\varepsilon >> 1 \Rightarrow e^{-\beta\varepsilon} \approx 0$

$\therefore \lim_{T\to 0} P_0 = 1$

Se $T \to \infty$ então $\beta\varepsilon \approx 0 \Rightarrow e^{-\beta\varepsilon} \approx 1$

$\therefore \lim_{T\to \infty} P_0 = \frac{1}{3}$



(c) $U = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta} \ln(1 + 2e^{-\beta\varepsilon})$

$$= \frac{2\varepsilon e^{-\beta\varepsilon}}{1 + 2e^{-\beta\varepsilon}}$$

$$\therefore \quad U = \frac{2\varepsilon}{e^{\beta\varepsilon} + 2} \quad //$$

$$S = \frac{U - F}{T} = \frac{2\varepsilon/T}{e^{\beta\varepsilon} + 2} + k_B \ln(1 + 2e^{-\beta\varepsilon})$$

$$\therefore \quad \frac{S}{k_B} = \frac{2\beta\varepsilon}{e^{\beta\varepsilon} + 2} + \ln(1 + 2e^{-\beta\varepsilon}) \quad //$$

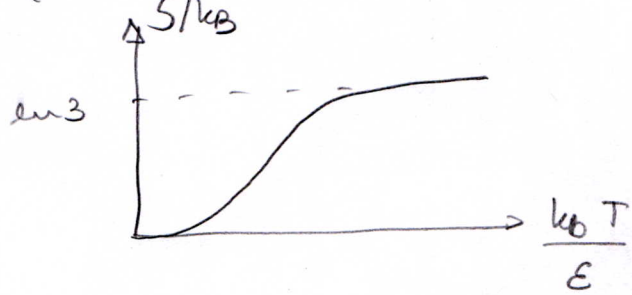
Se $T \to 0 \Rightarrow e^{-\beta\varepsilon} \simeq 0$ (ou $e^{\beta\varepsilon} \gg 1$)

$$\therefore \quad U \simeq 2\varepsilon e^{-\beta\varepsilon} \to 0$$

$$\frac{S}{k_B} \simeq \ln(1) = 0 \quad \leftarrow 3^{a} \text{ lei!}$$

Se $T \to \infty \Rightarrow e^{-\beta\varepsilon} \simeq 1$

$$\therefore \quad U \simeq \frac{2\varepsilon}{3}$$

$$\frac{S}{k_B} \simeq \ln 3 \quad //$$

(d)