Quantum Information and Quantum Noise - Problem set 2

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Deadline:

1. **Master equation for a qubit.** Use the Nakajima-Zwanzig method to derive the master equation for a qubit coupled to an infinite number of bosonic modes through the Hamiltonian
$$H = \frac{\omega}{2}\sigma_z + \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k(\sigma_+ b_k + \sigma_- b_k^\dagger). \tag{1}$$
This derivation is quite analogous to the one we did in class for the harmonic oscillator. I recommend you try to follow those steps.

2. **Generation of two-mode squeezing.** Consider two bosonic modes a and b. We define the two-mode squeezing operator as
$$S_z = \exp\Big\{z^* a^\dagger b^\dagger - zab\Big\}, \qquad z = re^{i\theta} \tag{2}$$
This is analogous in spirit to the single-mode squeezing, except that now it mixes a and b:
$$S_z^\dagger a S = a\cosh(r) + b^\dagger e^{i\theta}\sinh(r) \tag{3}$$
$$S_z^\dagger b S = b\cosh(r) + a^\dagger e^{i\theta}\sinh(r) \tag{4}$$
   (a) Find the covariance matrix Θ of the two-mode squeezed vacuum
$$\rho = S_z|0\rangle\langle 0|S_z^\dagger. \tag{5}$$
   Show from the CM that the reduced states of modes a and b are actually thermal states and relate the Bose-Einstein thermal occupation with the parameter z. Thus, we can think about a thermal state of a single mode as actually being a pure state of a larger system. This is the idea behind a theory called **thermal quantum field theory** developed to deal with quantum many-body systems at finite temperatures. However, this idea didn't really catch and another approach, due to Matsubara, became the dominant one. So people eventually forgot about it. But there is still a Wikipedia page!

   (b) Compute the Rényi-2 entanglement entropy of the two-mode squeezed vacuum as a function of z (Note: when I say "entanglement entropy" I already mean the entropy of the reduced states; of course, the entropy of ρ itself will be zero since it is a pure state).

3. **Generating two-mode squeezing using an open quantum system.** Let us now consider how two-mode squeezing may be generated. We consider a cavity populated with the two-modes a and b, which are subject to the Hamiltonian
$$H = \omega(a^\dagger a + b^\dagger b) + i\lambda(a^\dagger b^\dagger - ab),$$
(the factor of i is placed only for convenience). We also assume that the two modes are subject to a cavity loss dissipator, so that the total density matrix will evolve according to the master equation
$$\frac{d\rho}{dt} = -i[H,\rho] + 2\kappa D[a] + 2\kappa D[b],$$
where $D[L] = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$.

(a) Construct the Lyapunov equation for this problem and find the steady-state (possibly using the `LyapunovSolve[]` function in Mathematica).

(b) Discuss your result. Is the steady-state pure? What happens when $\lambda \to 0$? What about $\gamma \to 0$?

(c) Compute the Rényi-2 mutual information and make some pretty plots.

(d) Study the Duan inequality for your steady-state.

4. **Purity of a Gaussian state.** Consider a single bosonic mode with annihilation operator a. The most general Gaussian state is a displaced squeezed thermal state

$$\rho = D(\alpha) S_z \rho_{\text{th}} S_z^\dagger D^\dagger(\alpha), \qquad S_z = \exp\left\{\frac{1}{2}(z a^\dagger a^\dagger - z^* aa)\right\},$$

where $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ is the displacement operator, $S_z = \exp\left\{\frac{1}{2}(z a^\dagger a^\dagger - z^* aa)\right\}$ is the squeezing operator and ρ_{th} is the thermal density matrix $\rho_{\text{th}} = (1 - e^{-\beta\omega})e^{-\beta\omega a^\dagger a}$.

Show that the purity of this state is

$$\text{tr}(\rho^2) = \frac{1}{2\bar{n} + 1}$$

where $\bar{n} = (e^{\beta\omega} - 1)^{-1}$. Tip: exploit the fact that $D(\alpha)$ and S_z are unitary to get rid of them. Relate this result with the formula discussed in the lecture notes,

$$\text{tr}(\rho^2) = \frac{1}{2\sqrt{|\Theta|}},$$

where Θ is the covariance matrix.

5. **Optical Kerr bistability.** Consider the model described by the Hamiltonian

$$H = \omega_c a^\dagger a + \frac{U}{2} a^\dagger a^\dagger aa + i\epsilon(a^\dagger e^{-i\omega_p t} - a e^{i\omega_p t})$$

and subject to the cavity loss dissipator $D(\rho) = 2\kappa[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}]$. This roughly describes a optical cavity with a non-linear medium inside, which therefore generate a photon-photon interaction (the U term) (c.f. `arXiv 1608.00717` for a cool recent paper about it). It is also somewhat analogous to the types of interactions appearing in the Bose-Hubbard model which presents a quantum phase transition between a superfluid phase and a Mott insulator. This is an important model for ultra-cold atoms (c.f. Greiner *et. al.*, *Nature* **415** 39-44 (2002)).

(a) Find the equations of motion for $\langle a \rangle$. It will depend on higher order moments.

(b) Truncate it assuming that the pump is sufficiently large so that we can ignore the fluctuations.

(c) Analyze the steady-state by plotting $n = |\alpha|^2$ as a function of the pump ϵ. Show that for negative detuning, $\Delta = \omega_c - \omega_p < 0$, the system presents a bistable region where two steady-states can coexist.

(d) Actually, I lied. You will find three steady-states. But one of them is unstable. Optional: analyze the stability of the system by linearizing the ordinary differential equations.